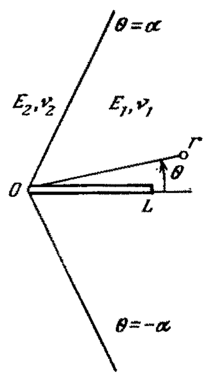
PMM U.S.S.R.,Vol.50,No.2,pp.252-254,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

A PIECEWISE-HOMOGENEOUS PLANE WITH A SEPARATING BOUNDARY IN THE FORM OF THE SIDES OF AN ANGLE AND A SYMMETRICAL CUT ORIGINATING FROM THE VERTEX*

L.A. KIPNIS

A plane problem of the equilibrium of an elastic plane formed by two different half-planes joined together along a boundary in the form of two sides of an angle, is considered. A symmetrical cut originates at the apex and a known, constant normal load is applied to its edges. A functional Wiener-Hopf equation of the problem is given and its exact closed solution is constructed. The stress intensity coefficient at the cut end is determined.



Let us consider the equilibrium of an elastic plane consisting of two different half-planes coupled rigidly along a boundary representing a set of rays $\theta = \pm\alpha, 0 < \alpha < \pi$ (see the figure). Young's moduli and Poisson's ratio of the right and left half-plane are, respectively, E_1, ν_1 and E_2, ν_2. When $\theta = 0, r < l$, we have a cut, with given constant normal loads applied to its edges. The stresses vanish at infinity. We require to find the stress intensity coefficient at the right end of the cut.

Confirming ourselves to considering the half-plane $0 \leqslant \theta \leqslant \pi$, we can write the boundary conditions for the symmetrical problem formulated in the form

$$\theta = \alpha, \ [\sigma_\theta] = [\tau_{r\theta}] = 0, \ [u_\theta] = [u_r] = 0 \tag{1}$$

$$\theta = \pi, \ \tau_{r\theta} = 0, \ u_\theta = 0; \ \theta = 0, \ \tau_{r\theta} = 0$$

$$\theta = 0, \ r < l, \ \sigma_\theta = -\sigma; \ \theta = 0, \ r > l, \ u_\theta = 0 \tag{2}$$

Here $\sigma_\theta, \tau_{r\theta}, \sigma_r$ are the stresses, u_θ, u_r the displacements, $[N]$ is a discontinuity of magnitude N, and σ is a given quantity.

When $r \to 0$, we have an asymptotic form representing a solution of the problem of a piecewise-homogeneous plane with a symmetrical semi-infinite cut emerging from the apex O, whose edges are stress-free. The solution is constructed using the method of singular solutions /1/. In particular we have

$$0 < \theta < \pi, \ r \to 0, \ \sigma_\theta = O(r^\lambda), \ \tau_{r\theta} = O(r^\lambda), \ \sigma_r = O(r^\lambda)$$

Here λ is a root of the equation

$$D(-\lambda - 1) = 0$$

$$D = \Delta d_1 k^2 - [\Delta d_2 - 4\delta\Delta_2 - (1+\varkappa_1)(1+\varkappa_2)\sin 2z\pi]k - 4\delta\Delta_2$$

$$\Delta = \sin 2z(\pi - \alpha) - z\sin 2\alpha, \quad \Delta_2 = \varkappa_2 \sin 2z(\pi - \alpha) + z\sin 2\alpha$$

$$\delta = \sin^2 z\alpha - z^2\sin^2\alpha, \quad d_1 = (1+\varkappa_1)^2 - 4(\varkappa_1\sin^2 z\alpha + z^2\sin^2\alpha)$$

$$d_2 = (1+\varkappa_1)(1+\varkappa_2) - 4(\varkappa_1\sin^2 z\alpha + z^2\sin^2\alpha)$$

$$k = \frac{E_2(1+\nu_1)}{E_1(1+\nu_2)}, \quad \varkappa_j = 3 - 4\nu_j \quad (j = 1, 2)$$

unique in the interval $(-1, 0)$.

When $\alpha = \pi/2$ the above equation becomes identical with the characteristic equation of the Zak-Williams problem /1, 2/ of a crack perpendicular to the boundary separating the different elastic media.

Applying the integral Mellin transform with a complex parameter p /3/ to the equations of equilibrium, the condition of compatibility of deformation, Hooke's law and conditions (1) and taking into account conditions (2), we arrive at the following functional Wiener-Hopf equation:

$$\Phi^-(p) = \operatorname{tg} p\pi\, G(p)\,[\Phi^+(p) + g(p)], \ g(p) = -\sigma/(p+1) \tag{3}$$

$$(-\varepsilon_1 < \operatorname{Re} p < \varepsilon_2, \ 0 < \varepsilon_{1,2} < 1)$$

$$G(p) = 2\,\frac{\Delta\Delta_1 k^2 - [\Delta\Delta_1 + \Delta_0\Delta_2 - (1+\varkappa_1)(1+\varkappa_2)\sin^2 p\pi]k + \Delta_0\Delta_2}{D(p)\operatorname{tg} p\pi}$$

$$\Delta_0 = \sin 2p\alpha + p\sin 2\alpha, \quad \Delta_1 = \varkappa_1 \sin 2p\alpha - p\sin 2\alpha$$

$$\Phi^+(p) = \int_1^\infty \sigma_\theta(\rho l, 0)\,\rho^p\,d\rho, \qquad \Phi^-(p) = \frac{E_1}{2(1-\nu_1^2)}\int_0^1 \left.\frac{\partial u_\theta}{\partial r}\right|_{\substack{r=\rho l\\ \theta=0}} \rho^p\,d\rho$$

*Prikl.Matem.Mekhan.,50,2,334-336,1986

Since $u_\theta(r, 0) \to 0$ as $r \to l$ and $r \to 0$, we have

$$\Phi^-(0) = 0 \tag{4}$$

Using the factorization

$$G(p) = G^+(p)/G^-(p) \quad (\operatorname{Re} p = 0) \tag{5}$$

$$\exp\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(t)}{t-p}\,dt\right] = \begin{cases} G^+(p), & \operatorname{Re} p < 0 \\ G^-(p), & \operatorname{Re} p > 0 \end{cases}$$

$$p \operatorname{ctg} p\pi = K^+(p)\,K^-(p), \quad K^\pm(p) = \Gamma(1 \mp p)/\Gamma(1/2 \mp p) \tag{6}$$

($\Gamma(z)$ is the Euler gamma function), we can rewrite (3) as

$$\frac{p\Phi^+(p)\,G^+(p)}{K^+(p)} - \frac{\sigma p G^+(p)}{(p+1)\,K^+(p)} = \Phi^-(p)\,K^-(p)\,G^-(p) \quad (\operatorname{Re} p = 0) \tag{7}$$

Using the representation

$$\frac{G^+(p)}{(p+1)\,K^+(p)} = \frac{1}{p+1}\left[\frac{G^+(p)}{K^+(p)} - \frac{G^+(-1)}{K^+(-1)}\right] + \frac{G^+(-1)}{(p+1)\,K^+(-1)}$$
$$(\operatorname{Re} p = 0)$$

we obtain, in accordance with (7),

$$\frac{p\Phi^+(p)\,G^+(p)}{K^+(p)} - \frac{\sigma p}{p+1}\left[\frac{G^+(p)}{K^+(p)} - \frac{G^+(-1)}{K^+(-1)}\right] = \tag{8}$$
$$\frac{\sigma p G^+(-1)}{(p+1)\,K^+(-1)} + \Phi^-(p)\,K^-(p)\,G^-(p) \quad (\operatorname{Re} p = 0)$$

The function on the left-hand side of (8) is analytic in the half-plane $\operatorname{Re} p < 0$, and the function on its right-hand side is analytic in the half-plane $\operatorname{Re} p > 0$. Therefore both these functions are equal to a single function analytic in the whole plane p.

Using the well-known asymptotic expression for an elastic field near the right end of a cut /1/, we find $(p \to \infty)$ that

$$\Phi^+(p) \sim K_I/\sqrt{-2pl}, \quad \Phi^-(p) \sim -K_I/\sqrt{2pl} \tag{9}$$

(K_I is the stress intensity coefficient at the right end of the cut).

From (5), (6), (9) it follows that the functions on the left and right side of (8) tend, as $p \to \infty$, to the constant

$$c = \sigma G^+(-1)/K^+(-1) - K_I/\sqrt{2l}$$

By Liouville's theorem the only analytic function is identically equal to a constant c over the whole plane p. In particular, $\Phi^-(0)K^-(0)G^-(0) = c$. According to (4), the last equation implies that $c = 0$. We write the solution of the functional equation as follows:

$$\Phi^+(p) = \frac{\sigma}{p+1}\left[\frac{G^+(p)}{K^+(p)} - \frac{G^+(-1)}{K^+(-1)}\right]\frac{K^+(p)}{G^+(p)} \quad (\operatorname{Re} p < 0)$$

$$\Phi^-(p) = -\frac{\sigma p G^+(-1)}{(p+1)\,K^+(-1)\,K^-(p)\,G^-(p)} \quad (\operatorname{Re} p > 0)$$

Using this solution and the inversion formula, we can find the stresses and displacements in the problem in question.

From the last formula we obtain $(p \to \infty)$

$$\Phi^-(p) \sim -G^+(-1)\,\sigma\sqrt{\pi/2}\,/\sqrt{2p} \tag{10}$$

According to (9), (10),

$$K_I = G^+(-1)\,\sigma\sqrt{\pi l/2} \tag{11}$$

The values of the function $G^+(-1)$ for $\nu_1 = \nu_2 = 1/3$ and some values of k and α are given in the table below

α, deg.	$k = 0{,}001$	0.1	3	5	100
15	8,663	2,450	0,683	0,587	0,376
30	4,345	2,034	0,750	0,675	0,523
45	2,860	1,735	0,807	0,750	0,642
60	2,168	1,521	0,856	0,814	0,739
90	1,541	1,256	0,929	0,909	0,876
120	1,311	1,124	0,971	0,963	0,950
150	1,234	1,055	0,991	0,988	0,982
165	1,209	1,028	0,996	0,994	0,990

If $E_1 = E_2$, $\nu_1 = \nu_2$ (a homogeneous plane with a cut), then from (3) it follows that $G(p) \equiv 1$ and (11) becomes identical with the well-known result for a Griffith crack.

In the same manner we solve the problem for the case when the crack edges are stress-free and the given asymptotic form representing the solution of the problem of a piecewise-homogeneous plane without a cut is realized at infinity. The latter solution is constructed using the method of singular solutions, and in this case the Wiener-Hopf equation differs from (3) in the form of the function $g(p)$ only.

REFERENCES

1. CHEREPANOV G.P., Mechanics of Brittle Fracture. Moscow, Nauka, 1974.
2. ZAK A.R. and WILLIAMS M.L., Crack point stress singularities at bimaterial interface. J. Appl. Mech., 30, 1, 1963.
3. UFLYAND YA.S., Integral Transforms in Problems of the Theory of Elasticity. Leningrad, Nauka, 1967.
4. GAKHOV F.D., Boundary Value Problems. Moscow, Nauka, 1977.
5. NOBLE B., Methods Based on the Wiener-Hopf Technique for Solving Partial Differential Equations. N.Y. Pergamon Press, 1958.

Translated by L.K.